ON THE HÖRMANDER CLASSES OF BILINEAR PSEUDODIFFERENTIAL OPERATORS

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ABSTRACT. Bilinear pseudodifferential operators with symbols in the bilinear analog of all the Hörmander classes are considered and the possibility of a symbolic calculus for the transposes of the operators in such classes is investigated. Precise results about which classes are closed under transposition and can be characterized in terms of asymptotic expansions are presented. This work extends the results for more limited classes studied before in the literature and, hence, allows the use of the symbolic calculus (when it exists) as an alternative way to recover the boundedness on products of Lebesgue spaces for the classes that yield operators with bilinear Calderón-Zygmund kernels. Some boundedness properties for other classes with estimates in the form of Leibniz' rule are presented as well.

1. INTRODUCTION

Many linear operators encountered in analysis are best understood when represented as singular integral operators in the space domain, while others are better treated as pseudo-differential operators in the frequency domain. In some particular situations both representations are readily available. In many others, however, one of them is only given abstractly, and through the existence of a distributional kernel or symbol which is hard or impossible to compute. Both representations have proved to be tremendously useful. The representation of operators as pseudodifferential ones usually yields simple L^2 estimates, explicit formulas for the calculus of transposes and composition, and invariance properties under change of coordinates in smooth situations. As it is known, this makes pseudodifferential operators an invaluable tool in the study of partial differential equations and they are employed to construct parametrices and study regularity properties of solutions. The integral representation on the other hand, is often best suited for other L^p estimates and motivates or indicates what results should hold in other metric and measure theoretic situations where the Fourier transform is no longer available. This has found numerous applications in complex analysis, operator theory, and also in problems in partial differential equations where the domains or functions involved have a minimum amount of regularity.

Work on singular integral and pseudodifferential operators started with explicit classical examples and was then directed to attack specific applications in other areas. Switching back and forth, many efforts where also oriented to the understanding of naturally appearing technical questions and to the testing of the full power of new techniques as they developed. In fact, sometimes the technical analytic tools studied preceded the applications in which they were much later used. The Calderón-Zygmund theory and the related real variable techniques played a tremendous role in all these accomplishments. This is explained in detail from both the historical and technical points of view in, for example, the book by Stein [28].

The study of bilinear operators within harmonic analysis is following a similar path. The first systematic treatment of bilinear singular integrals and pseudodifferential operators in the early work of Coifman and Meyer [13], [15] originated from specific problems about Calderón's commutators, and soon lead to the study of general boundedness properties of pseudodifferential operators [14]. Later on, the work of Lacey and Thiele [22], [23] on a specific singular integral operator (the bilinear Hilbert transform which also goes back to Calderón) and the new techniques developed by them immediately suggested the study of other bilinear operators and the need to understand and characterize their boundedness and computational properties. See for example Gilbert and Nahmod [16], Muscalu et al. [25], Grafakos and Li [17], Bényi et al. [3], to name a few. The development of the symbolic calculus for bilinear pseudodifferential operators started in Bényi and Torres [7] and was continued in Bényi et al. [5]. Other results specific to bilinear pseudodifferential operators were obtained in Bényi et al. [8], [2], [4], [6] and, much recently, in Bernicot [9] and Bernicot and Torres [10]. As in the linear case, many of the results obtained were motivated too by the Calderón-Zygmund theory and its bilinear counterpart as developed in Grafakos and Torres [18]; see also Christ and Journé [12], Kenig and Stein [19], Maldonado and Naibo [24]. The literature is by now vast, see [27] for further references.

We want to contribute with this article to the understanding of the properties of *all* the bilinear analogs of the linear Hörmander classes of pseudodifferential operators. These bilinear classes are denoted by $BS^m_{\rho,\delta}$ (see the next section for technical definitions). Only some particular cases of them have been studied before; mainly the cases when $\rho = 1$ or when $\rho = \delta = 0$. The symbolic calculus has only been developed for the case $\rho = 1$ and $\delta = 0$. Our goal is to complete the symbolic calculus for all the possible (and meaningful) values of δ and ρ.

We could quote from the introduction of Hörmander's work [20]: "In this work the use of Fourier transformations has been emphasized; as a result no singular integral operators are apparent...", but we are clearly guided by previous works that relate, in the case of operators of order zero, to bilinear Calderón-Zygmund singular integrals. In fact, the existence of calculus for $BS^0_{1,\delta}$, $\delta < 1$, gives an alternative way to prove the boundedness of such operators in the optimal range of L^p spaces directly from the multilinear $T1$-Theorem in [18]. That is, without using Littlewood-Paley arguments as in the already cited monograph [14] or the work [2].

While the composition of pseudodifferential operators (with linear ones) forces one to study different classes of operators introduced in [5], previous results in the subject left some level of uncertainty about whether the computation of transposes could still be accomplished within some other bilinear Hörmander classes. The forerunner work [7] dealt mainly with the class $BS_{1,0}^0$ and a significant part of the proofs given in [7] relied on the so-called Peetre inequality which does not go through in general for other values of $\rho \neq 1, \delta \neq 0, m \neq 0$. We resolve this problem in the present article using ideas inspired in part by some computations in Kumano-go [21], and developing the calculus of transposes for all the bilinear Hörmander classes for which such calculus is possible. The excluded classes are the ones for which $\rho = \delta = 1$. This restriction is really necessary as proved in [7]. In fact, as the linear class $S_{1,1}^0$, the class $BS_{1,1}^0$ is forbidden in the sense that two related pathologies occur: it is not closed under transposition and it contains operators which fail to be bounded on product of L^p spaces, even though the associated kernels for operators in this class are of bilinear Calderón-Zygmund type.

The analogy between results in the linear and multilinear situations is in general only a guide to what could be expected to transfer from one context to the other. Some multilinear results arise as natural counterparts to linear ones, but often the techniques employed need to be substantially sharpened or replaced by new ones. It is actually far more complicated to prove the existence of calculus in the bilinear case than in the linear one. Some properties of the symbols of bilinear pseudodifferential operators on $\mathbb{R}^n$ can be guessed from those of linear operators in $\mathbb{R}^{2n}$. Though some of our computations are reminiscent of those for linear pseudodifferential operators or Fourier integral operators, the calculus of transposes for bilinear operators does not follow from the linear results by doubling the number of dimensions. Boundedness results cannot be obtained in this fashion either. The essential obstruction is the fact that the integral of a function of two n-dimensional variables $(x, y) \in \mathbb{R}^{2n}$ yields no information about the (n-dimensional) integral of its restriction to the diagonal $(x, x), x \in \mathbb{R}^n$. On the other hand, a few *point-wise* estimates can be obtained in a more direct way from the linear case using the method of doubling the dimensions. For example, it is useful to establish first precise point-wise estimates on the bilinear kernels associated to the operators in various Hörmander classes. We are able to derive them from the linear ones investigated by Álvarez-Hounie [1].

It is interesting too that some results do not extend to the multilinear context. A notorious example is the Calderón-Vaillancourt result in [11] for the L^2 boundedness of the class $S_{0,0}^0$. One may expect the class $BS_{0,0}^0$ to map, say, $L^2 \times L^2$ into L^1, but this fails unless additional conditions on the symbol are imposed; see [8]. Such class only maps into an optimal modulation space $(M^{1,\infty})$ which is larger than L^1. See also [4] and [6] for more details. Similarly (and using duality and the existence of the calculus for transposes), it is natural to ask whether the class $BS_{\rho,\delta}^0$, with $0 < \delta < \rho < 1$ maps $L^2 \times L^\infty$ into L^2 – recall that Hörmander's results in [20] give that $S_{\rho,\delta}^0$ maps L^2 into L^2 for the same range of ρ and δ. Alternatively, it may only be possible to obtain the boundedness from $L^2 \times X$ into L^2, where X is a space smaller than L^∞. Though we do not know the answer to the former questions, we give a result in the direction of

the latter. We also obtain some other new boundedness properties involving Sobolev spaces which take the form of fractional Leibniz' rules.

In the next section we present the technical definitions and the precise statements of our results about symbolic calculus. Sections 3 and 4 contain the proofs of those results. Section 5 contains the results about the point-wise estimates for the kernels, while Section 6 has the boundedness results alluded to before.

2. Symbolic calculus in the bilinear Hörmander classes

We start by recalling the linear pseudodifferential operators in the general Hörmander classes $S_{\rho,\delta}^m$. These are operators of the form

$$T_\sigma(f)(x) = \int_{\mathbb{R}^n} \sigma(x,\xi)\widehat{f}(\xi)e^{ix\cdot\xi}\,d\xi$$

where the symbol σ satisfies the estimates

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \leq C_{\alpha\beta}(1+|\xi|)^{m+\delta|\alpha|-\rho|\beta|},$$

for all $x, \xi \in \mathbb{R}^n$, all multi-indices α, β, and some positive constants $C_{\alpha\beta}$. These operators are a priori defined for appropriate test functions.

In this article we study the natural bilinear analog

$$T_\sigma(f,g)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sigma(x,\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{ix\cdot(\xi+\eta)}\,d\xi d\eta,$$

where the symbol σ satisfies now the estimates

$$(2.1) \qquad |\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)| \leq C_{\alpha\beta\gamma}(1+|\xi|+|\eta|)^{m+\delta|\alpha|-\rho(|\beta|+|\gamma|)},$$

also for all $x, \xi, \eta \in \mathbb{R}^n$, all multi-indices α, β, γ and some positive constants $C_{\alpha\beta\gamma}$. The class of all symbols satisfying (2.1) is denoted by $BS_{\rho,\delta}^m(\mathbb{R}^n)$, or simply $BS_{\rho,\delta}^m$ when it is clear from the context to which space the variables x, ξ, η belong to.

The transposes of such operators are defined as usual by the duality relations

$$\langle T(f,g), h \rangle = \langle T^{*1}(h,g), f \rangle = \langle T^{*2}(f,h), g \rangle.$$

We will write

$$\sigma \sim \sum_{j=0}^\infty \sigma_j$$

if there is a non-increasing sequence $m_N \searrow -\infty$ such that

$$\sigma - \sum_{j=0}^{N-1} \sigma_j \in BS_{\rho,\delta}^{m_N},$$

for all $N > 0$.

The spaces of test functions that we will use will be the space C_c^∞ of infinitely differentiable functions with compact support or the Schwartz space $\mathcal{S}$. When given

their usual topologies, their duals are $\mathcal{D}'$ and $\mathcal{S}'$, the spaces of distributions and of tempered distributions, respectively. We will also consider C_c^s, $s \in \mathbb{N}$, the space of functions with compact support and continuous derivatives up to order s; $W^{s,2}$, the Sobolev space of functions having derivatives in L^2 up to order s; and $W_0^{s,\infty}$, the completion of C_c^s with respect to the norm $\sup_{|\gamma| \leq s} \|D^\gamma g\|_{L^\infty}$. Unless specified otherwise, the underlying space will be assumed to be $\mathbb{R}^n$.

We develop a symbolic calculus for bilinear pseudodifferential operators with symbols in all the bilinear Hörmander classes $BS_{\rho,\delta}^m$, $m \in \mathbb{R}$, $0 \leq \delta \leq \rho \leq 1$, $\delta < 1$.

Our first two theorems state that the Hörmander classes are closed under transposition and that the symbols of the transposed operators have appropriate asymptotic expansions.

Theorem 1 (Invariance under transposition). *Assume that $0 \leq \delta \leq \rho \leq 1$, $\delta < 1$, and $\sigma \in BS_{\rho,\delta}^m$. Then, for $j = 1, 2$, $T_\sigma^{*j} = T_{\sigma^{*j}}$, where $\sigma^{*j} \in BS_{\rho,\delta}^m$.*

Theorem 2 (Asymptotic expansion). *If $0 \leq \delta < \rho \leq 1$ and $\sigma \in BS_{\rho,\delta}^m$, then σ^{*1} and σ^{*2} have asymptotic expansions*

$$\sigma^{*1} \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} \partial_x^\alpha \partial_\xi^\alpha (\sigma(x, -\xi - \eta, \eta))$$

and

$$\sigma^{*2} \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} \partial_x^\alpha \partial_\eta^\alpha (\sigma(x, \xi, -\xi - \eta)).$$

More precisely, if $N \in \mathbb{N}$ then

$$(2.2) \qquad \sigma^{*1} - \sum_{|\alpha| < N} \frac{i^{|\alpha|}}{\alpha!} \partial_x^\alpha \partial_\xi^\alpha (\sigma(x, -\xi - \eta, \eta)) \in BS_{\rho,\delta}^{m+(\delta-\rho)N}$$

and

$$(2.3) \qquad \sigma^{*2} - \sum_{|\alpha| < N} \frac{i^{|\alpha|}}{\alpha!} \partial_x^\alpha \partial_\eta^\alpha (\sigma(x, \xi, -\xi - \eta)) \in BS_{\rho,\delta}^{m+(\delta-\rho)N}.$$

In relation to asymptotic expansions we also prove the following two theorems.

Theorem 3. *Assume that $a_j \in BS_{\rho,\delta}^{m_j}$, $j \geq 0$ and $m_j \searrow -\infty$ as $j \to \infty$. Then, there exists $a \in BS_{\rho,\delta}^{m_0}$ such that $a \sim \sum_{j=0}^{\infty} a_j$. Moreover, if*

$$b \in BS_{\rho,\delta}^\infty = \underset{m}{\cup} BS_{\rho,\delta}^m \text{ and } b \sim \sum_{j=0}^{\infty} a_j,$$

then

$$a - b \in BS_{\rho,\delta}^{-\infty} = \underset{m}{\cap} BS_{\rho,\delta}^m.$$

Theorem 4. *Assume that $a_j \in BS^{m_j}_{\rho,\delta}, j \geq 0$ and $m_j \searrow -\infty$ as $j \to \infty$. Let $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ be such that*

$$(2.4) \qquad |\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma a(x,\xi,\eta)| \leq C_{\alpha\beta\gamma}(1+|\xi|+|\eta|)^\mu,$$

for some positive constants $C_{\alpha\beta\gamma}$ and $\mu = \mu(\alpha,\beta,\gamma)$. If there exist $\mu_N \to \infty$ such that

$$(2.5) \qquad |a(x,\xi,\eta) - \sum_{j=0}^N a_j(x,\xi,\eta)| \leq C_N(1+|\xi|+|\eta|)^{-\mu_N},$$

then $a \in BS^{m_0}_{\rho,\delta}$ and $a \sim \sum_{j=0}^\infty a_j$.

The continuity on Lebesgue spaces of bilinear pseudodifferential operators with symbols in the class $BS^0_{1,\delta}$ with $0 \leq \delta < 1$ has been intensely addressed in the literature. It is nowadays a well-known fact that the bilinear kernels associated to bilinear operators with symbols in $BS^0_{1,\delta}$, $0 \leq \delta < 1$, are bilinear Calderón-Zygmund operators in the sense of Grafakos and Torres [18]. Recall the following result ([18, Corollary 1]) which is an application of the bilinear $T1$-Theorem therein:

If T and its transposes, T^{*1} and T^{*2}, have symbols in $BS^0_{1,1}$, then they can be extended as bounded operators from $L^p \times L^q$ into L^r for $1 < p,q < \infty$ and $1/p+1/q = 1/r$.

As mentioned in the introduction, and since $BS^0_{1,\delta} \subset BS^0_{1,1}$, we can directly combine this result with Theorem 1 to recover the following optimal version of a known fact.

Corollary 5. *If σ is a symbol in $BS^0_{1,\delta}$, $0 \leq \delta < 1$, then T_σ has a bounded extension from $L^p \times L^q$ into L^r, for all $1 < p,q < \infty, 1/p + 1/q = 1/r$.*

3. Proofs of Theorem 1 and Theorem 2

In the following, we assume that the symbol σ has compact support (in all three variables x, ξ, and η) so that the calculations in the proofs of Theorem 1 and Theorem 2 are properly justified. All estimates are obtained with constants independent of the support of σ and an approximation argument can be used to obtain the results for symbols that do not have compact support; see [7] for further details regarding such an approximation argument.

We restrict the proofs of Theorem 1 and Theorem 2 to the first transpose of T_σ ($j = 1$). As in [7], we rewrite T_σ^{*1} as a compound operator. We have

$$T^{*1}(h,g)(x) = \int_y \int_\eta \int_\xi c(y,\xi,\eta)h(y)\widehat{g}(\eta)e^{-i(y-x)\cdot\xi}e^{ix\cdot\eta}d\xi d\eta dy,$$

where

$$c(y,\xi,\eta) = \sigma(y,-\xi-\eta,\eta).$$

Straightforward calculations show that c satisfies the same differential inequalities as σ. Indeed, by the Leibniz rule we can write

$$
\begin{aligned}
|\partial_y^\alpha \partial_\xi^\beta \partial_\eta^\gamma c(y, \xi, \eta)| &\lesssim \sum_{|\gamma_1|+|\gamma_2|=|\gamma|} |\partial_y^\alpha \partial_\xi^\beta \partial_{\eta 2}^{\gamma_1} \partial_{\eta 3}^{\gamma_2} \sigma(y, -\xi - \eta, \eta)| \\
&\lesssim \sum (1 + |\xi + \eta| + |\eta|)^{m+\delta|\alpha|-\rho(|\beta|+|\gamma_1|+|\gamma_2|)} \\
&\lesssim (1 + |\xi| + |\eta|)^{m+\delta|\alpha|-\rho(|\beta|+|\gamma|)},
\end{aligned}
$$

(3.1)

with constants of the form $\sum_{|\gamma_1|+|\gamma_2|=|\gamma|} C_{\alpha\beta\gamma_j} 2^{m+\delta|\alpha|+\rho(|\beta|+|\gamma|)}$.

By appropriately changing variables of integration the symbol of T^{*1} is given in terms of c by the following expressin:

$$
a(x, \xi, \eta) = \int \int c(x + y, z + \xi, \eta) e^{-iz \cdot y} \, dy dz. \tag{3.2}
$$

Proof of Theorem 1. We will use the representation (3.2) of a to show that $a \in BS_{\rho,\delta}^m$. By (3.1) and since $\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma a(x, \xi, \eta) = \int \int \partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma c(x + y, z + \xi, \eta) e^{-iz \cdot y} \, dy dz$, it is enough to work with $\alpha = \beta = \gamma = 0$. Our techniques are inspired in part by ideas in [21, Lemma 2.4, page 69].

In the following, fix $\xi \in \mathbb{R}^n$, $\eta \in \mathbb{R}^n$, and set $A := 1 + |\xi| + |\eta|$. We have to prove that

$$
|a(x, \xi, \eta)| \lesssim A^m
$$

with a constant independent of the support of σ.

Let $l_0 \in \mathbb{N}$, $2l_0 > n$. Writing

$$
e^{-iz \cdot y} = (1 + A^{2\delta} |y|^2)^{-l_0} (1 + A^{2\delta}(-\Delta_z))^{l_0} e^{-iz \cdot y},
$$

integration by parts gives

$$
a(x, \xi, \eta) = \int \int q(x, y, z, \xi, \eta) e^{-iz \cdot y} \, dy dz, \tag{3.3}
$$

where

$$
q(x, y, z, \xi, \eta) = \frac{(1 + A^{2\delta}(-\Delta_z))^{l_0} c(x + y, z + \xi, \eta)}{(1 + A^{2\delta} |y|^2)^{l_0}}.
$$

We now estimate $(-\Delta_y)^l q$ for $l \in \mathbb{N}$.

(3.4)
$$
\begin{aligned}
(-\Delta_y)^l q &= \sum_{\substack{|\alpha|=2l \\ \alpha_i \text{ even}}} C_\alpha \, \partial_y^\alpha q(x, y, z, \xi, \eta) \\
&= \sum_{\substack{|\alpha|=2l \\ \alpha_i \text{ even}}} \sum_{\beta \leq \alpha} C_{\alpha\beta} \, \partial_y^\beta \left((1 + A^{2\delta} |y|^2)^{-l_0} \right) \partial_y^{\alpha-\beta} \left((1 + A^{2\delta}(-\Delta_z))^{l_0} c(x + y, z + \xi, \eta) \right).
\end{aligned}
$$

Note that

$$
\left| \partial_y^\beta \left((1 + A^{2\delta} |y|^2)^{-l_0} \right) \right| \leq C_{\beta l_0} A^{\delta|\beta|} \left(1 + A^{2\delta} |y|^2 \right)^{-l_0}. \tag{3.5}
$$

Moreover, if $P_{l_0} = \{\gamma = (\gamma_1, \cdots, \gamma_n) : \gamma_i$ even and $|\gamma| = 2j, j = 0, \cdots, l_0\}$, then

$$(1 + A^{2\delta}(-\Delta_z))^{l_0} c(x + y, z + \xi, \eta) = \sum_{\gamma \in P_{l_0}} C_\gamma A^{\delta|\gamma|} \partial_\xi^\gamma c(x + y, z + \xi, \eta),$$

and therefore

$$|\partial_y^{\alpha-\beta} \left((1 + A^{2\delta}(-\Delta_z))^{l_0} c(x + y, z + \xi, \eta) \right)|$$

(3.6)
$$\leq \sum_{\gamma \in P_{l_0}} C_{\gamma\alpha\beta} A^{\delta|\gamma|} (1 + |z + \xi| + |\eta|)^{m+\delta(|\alpha|-|\beta|)-\rho|\gamma|}.$$

From (3.4), (3.5) and (3.6), we get

(3.7)
$$|(-\Delta_y)^l q| \lesssim$$
$$\left(1 + A^{2\delta} |y|^2\right)^{-l_0} \sum_{\substack{|\alpha|=2l \\ \alpha_i \text{ even}}} \sum_{\beta \leq \alpha} C_{\alpha\beta l_0} A^{\delta|\beta|} \sum_{\gamma \in P_{l_0}} C_{\gamma\alpha\beta} A^{\delta|\gamma|} (1 + |z + \xi| + |\eta|)^{m+\delta(|\alpha|-|\beta|)-\rho|\gamma|}.$$

Define the sets

$$\Omega_1 = \{z : |z| \leq \tfrac{A^\delta}{2}\}, \quad \Omega_2 = \{z : \tfrac{A^\delta}{2} \leq |z| \leq \tfrac{A}{2}\}, \quad \Omega_3 = \{z : |z| \geq \tfrac{A}{2}\}.$$

We then have

$$a(x, \xi, \eta) = \int_{\Omega_1} \int_y \cdots + \int_{\Omega_2} \int_y \cdots + \int_{\Omega_3} \int_y \cdots := I_1 + I_2 + I_3.$$

Note that

(3.8)
$$\frac{1}{2} A \leq 1 + |z + \xi| + |\eta| \leq \frac{3}{2} A, \qquad z \in \Omega_1 \cup \Omega_2.$$

and

(3.9)
$$1 + |z + \xi| + |\eta| \leq A + |z| \leq 3 |z|, \qquad z \in \Omega_3.$$

Estimation for I_1. The estimate (3.7) with $l = 0$, (3.8), and $\delta - \rho \leq 0$ give, for $z \in \Omega_1$,

$$|q| \leq (1 + A^{2\delta} |y|^2)^{-l_0} \sum_{\gamma \in P_{l_0}} C_\gamma A^{\delta|\gamma|} (1 + |z + \xi| + |\eta|)^{m-\rho|\gamma|}$$

$$\leq (1 + A^{2\delta} |y|^2)^{-l_0} \sum_{\gamma \in P_{l_0}} C_\gamma A^{m+(\delta-\rho)|\gamma|}$$

$$\lesssim (1 + A^{2\delta} |y|^2)^{-l_0} A^m.$$

Therefore, since $2l_0 > n$,

$$|I_1| \lesssim A^m \int_{\Omega_1} \int_y \frac{1}{(1 + A^{2\delta} |y|^2)^{l_0}} \, dy dz \sim A^m.$$

Estimation for I_2. Integration by parts gives

$$\int_y q(x,y,z,\xi,\eta)e^{-iz\cdot y}\,dy = \frac{1}{|z|^{2l_0}}\int_y q(x,y,z,\xi,\eta)(-\Delta_y)^{l_0}e^{-iz\cdot y}\,dy$$

$$= \frac{1}{|z|^{2l_0}}\int_y (-\Delta_y)^{l_0}(q(x,y,z,\xi,\eta))e^{-iz\cdot y}\,dy.$$

Using (3.7) with $l = l_0$, (3.8), and $\delta - \rho \le 0$ we get, for $z \in \Omega_2$,

$$\left|(-\Delta_y)^{l_0}q\right|$$

$$\le \left(1 + A^{2\delta}|y|^2\right)^{-l_0} \sum_{\substack{|\alpha|=2l_0 \\ \alpha_i\ \text{even}}} \sum_{\beta\le\alpha} C_{\alpha\beta l_0} A^{\delta|\beta|} \sum_{\gamma\in P_{l_0}} c_{\gamma\alpha\beta}A^{\delta|\gamma|}A^{m+\delta(|\alpha|-|\beta|)-\rho|\gamma|}$$

$$\le \left(1 + A^{2\delta}|y|^2\right)^{-l_0} \sum_{\substack{|\alpha|=2l_0 \\ \alpha_i\ \text{even}}} \sum_{\beta\le\alpha} C_{\alpha\beta l_0} \sum_{\gamma\in P_{l_0}} c_{\gamma\alpha\beta}A^{m+\delta|\alpha|+(\delta-\rho)|\gamma|}$$

$$\lesssim \frac{A^{m+2l_0\delta}}{\left(1 + A^{2\delta}|y|^2\right)^{l_0}}.$$

Recalling that $2l_0 > n$, we get

$$|I_2| \le \int_{\Omega_2} \frac{1}{|z|^{2l_0}}\int_y \frac{A^{m+2l_0\delta}}{\left(1 + A^{2\delta}|y|^2\right)^{l_0}}\,dy dz$$

$$\lesssim A^{m+2l_0\delta-\delta n}\int_{|z|\ge\frac{A^\delta}{2}} |z|^{-2l_0}\,dz \sim A^m.$$

Estimation for I_3. Let $l \in \mathbb{N}$ to be chosen later. Again, integration by parts gives

$$\int_y q(x,y,z,\xi,\eta)e^{-iz\cdot y}\,dy = \frac{1}{|z|^{2l}}\int_y q(x,y,z,\xi,\eta)(-\Delta_y)^l e^{-iz\cdot y}\,dy$$

$$= \frac{1}{|z|^{2l}}\int_y (-\Delta_y)^l(q(x,y,z,\xi,\eta))e^{-iz\cdot y}\,dy.$$

Using (3.7) and (3.9), and defining $m_+ = \max(0,m)$, we get, for $z \in \Omega_3$,

$$\left|(-\Delta_y)^l q\right|$$

$$\lesssim \left(1 + A^{2\delta}|y|^2\right)^{-l_0} \sum_{\substack{|\alpha|=2l \\ \alpha_i\ \text{even}}} \sum_{\beta\le\alpha} C_{\alpha\beta l_0} A^{\delta|\beta|} \sum_{\gamma\in P_{l_0}} C_{\gamma\alpha\beta}A^{\delta|\gamma|}(1+|z+\xi|+|\eta|)^{m+\delta(|\alpha|-|\beta|)-\rho|\gamma|}$$

$$\lesssim \left(1 + A^{2\delta}|y|^2\right)^{-l_0} \sum_{\substack{|\alpha|=2l \\ \alpha_i\ \text{even}}} \sum_{\beta\le\alpha} C_{\alpha\beta l_0} \sum_{\gamma\in P_{l_0}} C_{\gamma,\alpha,\beta}|z|^{\delta(|\beta|+|\gamma|)}|z|^{m_++\delta(|\alpha|-|\beta|)}$$

$$\lesssim \left(1 + A^{2\delta}|y|^2\right)^{-l_0}|z|^{m_++\delta(2l+2l_0)}.$$

We then have

$$|I_3| \lesssim \int_{\Omega_3} \frac{1}{|z|^{2l}} \int_y \left(1 + A^{2\delta} |y|^2\right)^{-l_0} |z|^{m_+ + \delta(2l + 2l_0)} \, dydz$$

$$\sim \int_{|z| \geq \frac{A}{2}} |z|^{m_+ + 2l_0\delta + 2l(\delta - 1)} \, dz \int_y \left(1 + A^{2\delta} |y|^2\right)^{-l_0} \, dy$$

$$\sim A^{-\delta n} \int_{|z| \geq \frac{A}{2}} |z|^{m_+ + 2l_0\delta + 2l(\delta - 1)} \, dz.$$

We now choose $l \in \mathbb{N}$ sufficiently large so that

$$m_+ + 2l_0\delta + 2l(\delta - 1) < -n \quad \text{and} \quad -\delta n + m_+ + 2l_0\delta + 2l(\delta - 1) + n < m.$$

The existence of such an l is guaranteed by the condition $0 \leq \delta < 1$.

Finally,

$$|I_3| \lesssim A^{-\delta n + m_+ + 2l_0\delta + 2l(\delta - 1) + n} \leq A^m.$$

$\square$

Proof of Theorem 2. As in the proof of Theorem 1 we use the representation (3.2) for the symbol of T^{*1}. Define

$$a_\alpha(x, \xi, \eta) := \frac{i^{|\alpha|}}{\alpha!} \partial_x^\alpha \partial_\xi^\alpha c(x, \xi, \eta) = \frac{1}{\alpha!} \int \int \partial_\xi^\alpha c(x + y, \xi, \eta) \, e^{-iz \cdot y} z^\alpha \, dydz.$$

By the estimates (3.1), $a_\alpha \in BS_{\rho,\delta}^{m+(\delta-\rho)|\alpha|}$ with constants independent of the support of σ. We will show that

(3.10)

$$\left| \partial_x^{\alpha_1} \partial_\xi^{\alpha_2} \partial_\eta^{\alpha_3} \left(a(x, \xi, \eta) - \sum_{|\alpha| < N} a_\alpha(x, \xi, \eta) \right) \right| \leq C (1 + |\xi| + |\eta|)^{m + (\delta - \rho)N + \delta|\alpha_1| - \rho(|\alpha_2| + |\alpha_3|)},$$

where $C = C_{\alpha_1 \alpha_2 \alpha_3 N}$ is independent of the support of σ. This then shows that $a - \sum_{|\alpha| < N} a_\alpha \in BS_{\delta,\rho}^{m+(\delta-\rho)N}$ and therefore we have (2.2). Now, by Taylor's theorem

$$a(x, \xi, \eta) - \sum_{|\alpha| < N} a_\alpha(x, \xi, \eta) = \sum_{|\alpha| = N} \frac{1}{\alpha!} \int \int \partial_\xi^\alpha c(x + y, \xi + tz, \eta) \, z^\alpha \, e^{-iz \cdot y} \, dydz,$$

where $t \in (0, 1)$ and $t = t(x, y, \xi, z, \eta)$. Note that, because of the estimates (3.1), it is enough to prove (3.10) for $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

Inequality (3.10) follows from computations similar to the ones in Theorem 1. We include them here for the reader's convenience and for completeness.

Fix $N \in \mathbb{N}_0$ and a multiindex α with $|\alpha| = N$. For $l_0 \in \mathbb{N}$, integration by parts gives

(3.11) $I_\alpha := \int \int \partial_\xi^\alpha c(x + y, \xi + tz, \eta) \, z^\alpha \, e^{-iz \cdot y} \, dydz = \int \int q(x, y, z, \xi, \eta) e^{-iz \cdot y} \, dydz,$

where

$$q(x, y, z, \xi, \eta) = \frac{(1 + A^{2\delta}(-\Delta_z))^{l_0} (\partial_\xi^\alpha c(x + y, \xi + tz, \eta) \, z^\alpha)}{(1 + A^{2\delta} |y|^2)^{l_0}}.$$

We now estimate $(-\Delta_y)^l q$ for $l \in \mathbb{N}$.

(3.12)
$$(-\Delta_y)^l q = \sum_{\substack{|\nu|=2l \\ \nu_i \text{ even}}} C_\nu \, \partial_y^\nu q(x, y, z, \xi, \eta)$$

$$= \sum_{\substack{|\nu|=2l \\ \nu_i \text{ even} \\ \beta \leq \nu}} C_{\nu\beta} \, \partial_y^\beta \left((1 + A^{2\delta} |y|^2)^{-l_0}\right) \partial_y^{\nu-\beta} \left((1 + A^{2\delta}(-\Delta_z))^{l_0} (\partial_\xi^\alpha c(x+y, \xi+tz, \eta) \, z^\alpha)\right).$$

As before, if $P_{l_0} = \{\gamma = (\gamma_1, \cdots, \gamma_n) : \gamma_i \text{ even and } |\gamma| = 2j, \; j = 1, \cdots, l_0 \}$, then

$$(1 + A^{2\delta}(-\Delta_z))^{l_0} (\partial_\xi^\alpha c(x+y, \xi+tz, \eta) \, z^\alpha) = \sum_{\gamma \in P_{l_0}} C_\gamma A^{\delta|\gamma|} \partial_z^\gamma (\partial_\xi^\alpha c(x+y, \xi+tz, \eta) \, z^\alpha)$$

$$= \sum_{\substack{\gamma \in P_{l_0} \\ \omega \leq \gamma, \omega \leq \alpha}} C_{\gamma\omega} \, A^{\delta|\gamma|} \, \partial_z^\omega z^\alpha \, (\partial_\xi^{\alpha+\gamma-\omega} c)(x+y, \xi+tz, \eta) \, t^{|\gamma|-|\omega|},$$

and therefore

$$\left|\partial_y^{\nu-\beta} \left((1 + A^{2\delta}(-\Delta_z))^{l_0} (\partial_\xi^\alpha c(x+y, \xi+tz, \eta) \, z^\alpha)\right)\right|$$

(3.13)
$$\leq \sum_{\substack{\gamma \in P_{l_0} \\ \omega \leq \gamma, \omega \leq \alpha}} C_{\gamma\omega\nu\beta} \, A^{\delta|\gamma|} \, |z|^{N-|\omega|} \, (1 + |\xi+tz| + |\eta|)^{m+\delta(|\nu|-|\beta|)-\rho(N+|\gamma|-|\omega|)}.$$

From (3.12), (3.5) and (3.13), we get

(3.14)
$$\left|(-\Delta_y)^l q\right| \lesssim (1 + A^{2\delta} |y|^2)^{-l_0}$$

$$\times \sum_{\substack{|\nu|=2l, \nu_i \text{ even} \\ \beta \leq \nu, \gamma \in P_{l_0} \\ \omega \leq \gamma, \omega \leq \alpha}} C_{\gamma\omega\nu\beta l_0} \, A^{\delta(|\beta|+|\gamma|)} \, |z|^{N-|\omega|} \, (1 + |\xi+tz| + |\eta|)^{m+\delta(|\nu|-|\beta|)-\rho(N+|\gamma|-|\omega|)}.$$

Letting again

$$\Omega_1 = \{z : |z| \leq \tfrac{A^\delta}{2}\}, \quad \Omega_2 = \{z : \tfrac{A^\delta}{2} \leq |z| \leq \tfrac{A}{2}\}, \quad \Omega_3 = \{z : |z| \geq \tfrac{A}{2}\},$$

we have

$$I_\alpha = \int_{\Omega_1} \int_y \cdots + \int_{\Omega_2} \int_y \cdots + \int_{\Omega_3} \int_y \cdots := I_1 + I_2 + I_3.$$

Note that

(3.15)
$$\frac{1}{2} A \leq 1 + |\xi+tz| + |\eta| \leq \frac{3}{2} A, \qquad z \in \Omega_1 \cup \Omega_2, \; t \in (0,1),$$

and

(3.16)
$$1 + |\xi+tz| + |\eta| \leq A + |z| \leq 3|z|, \qquad z \in \Omega_3, \; t \in (0,1).$$

The estimates in (3.10) for $\alpha_1 = \alpha_2 = \alpha_3 = 0$ follow if we prove that

$$|I_i| \lesssim A^{m+(\delta-\rho)N}, \qquad i = 1, 2, 3.$$

Estimation for I_1. The estimate (3.14) with $l = 0$, (3.15), and $\delta - \rho < 0$ give, for $z \in \Omega_1$,

$$|q| \leq (1 + A^{2\delta} |y|^2)^{-l_0} \sum_{\substack{\gamma \in P_{l_0} \\ \omega \leq \gamma, \omega \leq \alpha}} C_{\gamma \omega l_0} A^{m+(\delta-\rho)(|\gamma|+N-|\omega|)} \quad \lesssim \frac{A^{m+(\delta-\rho)N}}{(1 + A^{2\delta} |y|^2)^{l_0}}.$$

Therefore, if we choose $2l_0 > n$, we get

$$|I_1| \lesssim A^{m+(\delta-\rho)N} \int_{\Omega_1} \int_y \frac{1}{(1 + A^{2\delta} |y|^2)^{l_0}} \, dy dz \sim A^{m+(\delta-\rho)N}.$$

Estimation for I_2. Integration by parts gives

$$\int_y q(x, y, z, \xi, \eta) e^{-iz \cdot y} \, dy = \frac{1}{|z|^{2l_0}} \int_y q(x, y, z, \xi, \eta)(-\Delta_y)^{l_0} e^{-iz \cdot y} \, dy$$

$$= \frac{1}{|z|^{2l_0}} \int_y (-\Delta_y)^{l_0} (q(x, y, z, \xi, \eta)) e^{-iz \cdot y} \, dy.$$

Using (3.14) with $l = l_0$, (3.15), and $\delta - \rho < 0$ we get, for $z \in \Omega_2$,

$$\left| (-\Delta_y)^{l_0} q \right|$$
$$\leq \left(1 + A^{2\delta} |y|^2 \right)^{-l_0} \sum_{\substack{|\nu|=2l_0, \nu_i \text{ even} \\ \beta \leq \nu, \gamma \in P_{l_0} \\ \omega \leq \gamma, \omega \leq \alpha}} C_{\gamma \omega \nu \beta l_0} A^{\delta|\gamma|} |z|^{N-|\omega|} A^{m+\delta 2l_0 - \rho(N+|\gamma|-|\omega|)}.$$

Choosing l_0 such that $2l_0 > N + n$,

$$|I_2| \leq A^{-n\delta} \sum_{\substack{|\nu|=2l_0, \nu_i \text{ even} \\ \beta \leq \nu, \gamma \in P_{l_0} \\ \omega \leq \gamma, \omega \leq \alpha}} C_{\gamma \omega \nu \beta l_0} A^{\delta|\gamma|} A^{m+\delta 2l_0 - \rho(N+|\gamma|-|\omega|)} \int_{|z| \geq \frac{A^\delta}{2}} |z|^{N-|\omega|-2l_0} \, dz$$
$$\lesssim A^{m+(\delta-\rho)N}.$$

Estimation for I_3. Let $l \in \mathbb{N}$ to be chosen later. Again, integration by parts gives

$$\int_y q(x, y, z, \xi, \eta) e^{-iz \cdot y} \, dy = \frac{1}{|z|^{2l}} \int_y q(x, y, z, \xi, \eta)(-\Delta_y)^l e^{-iz \cdot y} \, dy$$

$$= \frac{1}{|z|^{2l}} \int_y (-\Delta_y)^l (q(x, y, z, \xi, \eta)) e^{-iz \cdot y} \, dy.$$

Using (3.14) and (3.16), and defining $m_+ = \max(0, m)$, we get, for $z \in \Omega_3$,

$$\left| (-\Delta_y)^l q \right| \lesssim \left(1 + A^{2\delta} |y|^2 \right)^{-l_0} \sum_{\substack{|\nu|=2l, \nu_i \text{ even} \\ \beta \leq \nu, \gamma \in P_{l_0} \\ \omega \leq \gamma, \omega \leq \alpha}} C_{\gamma \omega \nu \beta l_0} |z|^{\delta(|\beta|+2l_0)} |z|^{N-|\omega|} |z|^{m_+ + \delta(2l-|\beta|)}$$

We then have

$$|I_3| \lesssim A^{-\delta n} \sum_{\substack{|\nu|=2l, \nu_i \text{ even} \\ \beta \leq \nu, \gamma \in P_{l_0} \\ \omega \leq \gamma, \omega \leq \alpha}} C_{\gamma\omega\nu\beta l_0} \int_{|z| \geq \frac{A}{2}} |z|^{m_+ + 2l(\delta-1) + 2l_0\delta + N} \, dz.$$

Choosing $l \in \mathbb{N}$ sufficiently large so that

$$m_+ + 2l(\delta-1) + 2l_0\delta + N < -n \quad \text{and} \quad -\delta n + m_+ + 2l(\delta-1) + 2l_0\delta + N + n < m + (\delta-\rho)N,$$

we obtain

$$|I_3| \lesssim A^{-\delta n + m_+ + 2l(\delta-1) + 2l_0\delta + N + n} \leq A^{m+(\delta-\rho)N}.$$

$\square$

4. Proofs of Theorem 3 and Theorem 4

Proof of Theorem 3. The second part of the statement is immediate if we write for all $N > 0$,

$$a - b = \left(a - \sum_{j=0}^{N-1} a_j\right) - \left(b - \sum_{j=0}^{N-1} a_j\right) \in BS_{\rho,\delta}^{m_N},$$

and recall that $m_N \searrow -\infty$ as $N \to \infty$.

For the proof of the first part of Theorem 3 we proceed by explicitly constructing a. Let $\psi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that $0 \leq \psi \leq 1$, $\psi(\xi, \eta) = 0$ on $\{(\xi, \eta) : |\xi| + |\eta| \leq 1\}$ and $\psi(\xi, \eta) = 1$ on $\{(\xi, \eta) : |\xi| + |\eta| \geq 2\}$. Define

$$a(x, \xi, \eta) = \sum_{j=0}^{\infty} \psi(\epsilon_j \xi, \epsilon_j \eta) a_j(x, \xi, \eta),$$

where $\epsilon_j \searrow 0$ as $j \to \infty$ is an appropriately chosen sequence of numbers in $(0, 1)$ so that $a \in BS_{\rho,\delta}^{m_0}$. The choice of this sequence will be made explicit below.

For each fixed $\epsilon \in (0, 1)$ we have

(a) $\psi(\epsilon\xi, \epsilon\eta) = 0$ for $|\xi| + |\eta| \leq 1/\epsilon$;
(b) $\psi(\epsilon\xi, \epsilon\eta) = 1$ for $|\xi| + |\eta| \geq 2/\epsilon$;
(c) For all β, γ such that $|\beta| + |\gamma| \geq 1$, $\partial_\xi^\beta \partial_\eta^\gamma \psi(\epsilon\xi, \epsilon\eta) = 0$ for $|\xi| + |\eta| \leq 1/\epsilon$ or $|\xi| + |\eta| \geq 2/\epsilon$;
(d) $|\partial_\xi^\beta \partial_\eta^\gamma \psi(\epsilon\xi, \epsilon\eta)| \leq c_{\beta\gamma} \epsilon^{|\beta|+|\gamma|}$.

In particular, because of (a) and (b) (that is, we only care about pairs (ξ, η) such that $1/\epsilon < |\xi| + |\eta| < 2/\epsilon$), (d) is equivalent to

(e) $|\partial_\xi^\beta \partial_\eta^\gamma \psi(\epsilon\xi, \epsilon\eta)| \leq c_{\beta\gamma}(1 + |\xi| + |\eta|)^{-|\beta|-|\gamma|}$.

This in turn is equivalent to saying that the family $\{\psi(\epsilon\xi, \epsilon\eta)\}_{0<\epsilon<1}$ represents a bounded set in $BS_{1,0}^0$ (endowed with the topology induced by appropriate semi-norms; see [7]).

Based on the estimate (e) and $a_j \in BS_{\rho,\delta}^{m_j}$, we can control each of the terms in the sum that defines a. By using Leibniz' rule, we immediately obtain

$$(4.1) \qquad |\partial_\xi^\beta \partial_\eta^\gamma \psi(\epsilon_j \xi, \epsilon_j \eta) a_j(x, \xi, \eta)| \leq C_{j,\alpha\beta\gamma}(1 + |\xi| + |\eta|)^{m_j + \delta|\alpha| - \rho(|\beta| + |\gamma|)}.$$

Let us now select ϵ_j such that $C_{j,\alpha\beta\gamma}\epsilon_j \leq 2^{-j}$ for all $|\alpha + \beta + \gamma| \leq j$. Due to (b) and (4.1), we can therefore write, for all $|\alpha + \beta + \gamma| \leq j$,

$$(4.2) \qquad |\partial_\xi^\beta \partial_\eta^\gamma \psi(\epsilon_j \xi, \epsilon_j \eta) a_j(x, \xi, \eta)| \leq 2^{-j}(1 + |\xi| + |\eta|)^{m_j + 1 + \delta|\alpha| - \rho(|\beta| + |\gamma|)}.$$

Now, for a fixed (x, ξ, η), the sum defining $a(x, \xi, \eta)$ is finite. Indeed, by (a), if infinitely many terms corresponding to a subsequence (j_k) are non-zero, we necessarily have $|\xi| + |\eta| > 1/\epsilon_{j_k} \to \infty$ as $k \to \infty$, a contradiction. In particular, we also have $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$.

Fix then a triple of multi-indices (α, β, γ) and let $J > 0$ be such that $|\alpha + \beta + \gamma| \leq J$. We split

$$a = S_1(a) + S_2(a),$$

where

$$S_1(a) = \sum_{j=0}^{J-1} \psi(\epsilon_j \xi, \epsilon_j \eta) a_j(x, \xi, \eta)$$

and

$$S_2(a) = \sum_{j=J}^{\infty} \psi(\epsilon_j \xi, \epsilon_j \eta) a_j(x, \xi, \eta).$$

Since $S_1(a)$ is a finite sum and each of its terms belongs to $BS_{\rho,\delta}^{m_j} \subset BS_{\rho,\delta}^{m_0}$, we infer that $S_1(a) \in BS_{\rho,\delta}^{m_0}$.

To estimate $S_2(a)$, recall first that for all $j \geq J$, $m_j + 1 \leq m_J + 1 \leq m_0$, thus, by using (4.2), we get that

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma S_2(a)| \leq \left(\sum_{j=J}^{\infty} 2^{-j}\right)(1 + |\xi| + |\eta|)^{m_0 + \delta|\alpha| - \rho(|\beta| + |\gamma|)},$$

which implies that $S_2(a) \in BS_{\rho,\delta}^{m_0}$.

Thus, we conclude that $a \in BS_{\rho,\delta}^{m_0}$. We finally arrive to the asymptotic expansion of a. We have

$$a - \sum_{j=0}^{N-1} a_j = \sum_{j=0}^{N-1} (\psi(\epsilon_j \xi, \epsilon_j \eta) - 1)a_j + \sum_{j=N}^{\infty} \psi(\epsilon_j \xi, \epsilon_j \eta)a_j.$$

In the first sum, because of (b), we only care about $|\xi| + |\eta| < 2/\epsilon_{N-1}$, and therefore we can achieve whatever decay we wish. For the second sum, we proceed exactly as above to show that it belongs to $BS_{\rho,\delta}^{m_N}$. The proof is complete. $\square$

Proof of Theorem 4. Note that, by Theorem 3, we know that there exists some $b \in BS_{\rho,\delta}^{m_0}$ such that

$$b \sim \sum_{j=0}^{\infty} a_j.$$

Therefore, it will be sufficient to show that $a - b \in BS^{-\infty}$. We start by noticing that

$$|a(x, \xi, \eta) - b(x, \xi, \eta)|$$

$$\leq |a(x, \xi, \eta) - \sum_{j=0}^{N-1} a_j(x, \xi, \eta)| + |b(x, \xi, \eta) - \sum_{j=0}^{N-1} a_j(x, \xi, \eta)|$$

$$\leq C_N(1 + |\xi| + |\eta|)^{-\mu_{N-1}} + \tilde{C}_N(1 + |\xi| + |\eta|)^{m_N}$$

$$\leq c_N(1 + |\xi| + |\eta|)^{-N},$$

because both $-\mu_{N-1}$ and m_N converge to $-\infty$ as $N \to \infty$.

To estimate the derivatives of the difference $a - b$ we will employ the following useful result; see the book by Taylor [26], p.41:

If K_1, K_2 are two compact sets such that $K_1 \subset K_2^\circ \subset K_2$ and $u \in C_c^2(\mathbb{R}^n)$, then

$$\sum_{|\alpha|=1} \sup_{z \in K_1} |D^\alpha u(z)| \lesssim \sup_{z \in K_2} |u(z)| \sum_{|\alpha| \leq 2} \sup_{z \in K_2} |D^\alpha u(z)|.$$

This is an immediate consequence of the estimate of a first order partial differential operator in terms of its second order partial differential operator:

$$\|\partial u / \partial x_j\|_{L^\infty}^2 \lesssim \|u\|_{L^\infty} \|\partial^2 u / \partial x_j^2\|_{L^\infty}, \qquad u \in C_c^2(\mathbb{R}^n).$$

Let then K be a compact set such that $x \in K$. Set $K_1 = K \times \{0\} \times \{0\}$, and let K_2 be a compact neighborhood of K_1. For fixed ξ, η, define

$$F_{\xi,\eta}(x, \zeta, \zeta') = a(x, \xi + \zeta, \eta + \zeta') - b(x, \xi + \zeta, \eta + \zeta').$$

We can write

$$\sup_{x \in K} |\nabla_{x,\xi,\eta}(a - b)(x, \xi, \eta)|^2 = \sup_{(x,\zeta,\zeta') \in K_1} |\nabla_{(x,\zeta,\zeta')} F_{\xi,\eta}(x, \zeta, \zeta')|^2$$

$$\lesssim \sup_{(x,\zeta,\zeta') \in K_2} |F_{\xi,\eta}(x, \zeta, \zeta')| \sum_{|\alpha| \leq 2} \sup_{(x,\zeta,\zeta') \in K_2} |D_{x,\zeta,\zeta'}^\alpha (a - b)(x, \xi + \zeta, \eta + \zeta')|$$

$$\leq C_N \sup_{(x,\zeta,\zeta')} (1 + |\xi + \zeta| + |\eta + \zeta'|)^{-N}(1 + |\xi + \zeta| + |\eta + \zeta'|)^{\max(\mu, m_0 + 2(\delta - \rho))}.$$

Since we have the freedom of choosing the compact neighborhood K_2, we can assume that on it $|\zeta| \leq 1/3, |\zeta'| \leq 1/3$. Then, by the triangle inequality, we have

$$1 + |\xi + \zeta| + |\eta + \zeta'| \geq \frac{1}{3}(1 + |\xi| + |\eta|),$$

and therefore we get, for all $N > 0$, the estimate

$$|\partial_x \partial_\xi \partial_\eta (a - b)(x, \xi, \eta)| \leq C_N 3^N 2^{\max(\mu, m_0 + 2(\delta - \rho))}(1 + |\xi| + |\eta|)^{-N}.$$

Analogously, we will be able to control all the derivatives of $a - b$ by

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma (a - b)(x, \xi, \eta)| \leq C_N(\alpha, \beta, \gamma)(1 + |\xi| + |\eta|)^{-N}.$$

This proves that $a - b \in BS^{-\infty}$ and the proof is complete. $\qquad \square$

5. Pointwise kernel estimates.

In this section we will describe decay/blow-up estimates for bilinear kernels associated to pseudodifferntial operators. These estimates can be summarized as follows.

Theorem 6. *Let $p \in BS^0_{\rho,\delta}$, $0 < \rho \leq 1$, $0 \leq \delta < 1$, $m \in \mathbb{R}$, and let $k(x,y,z)$ denote the distributional kernel of associated bilinear pseudodifferential operator T_p. Let $\mathbf{Z}^+$ denote the set of non-negative integers and for $x,y,z \in \mathbb{R}^n$, set*

$$S(x,y,z) = |x-y| + |x-z| + |y-z|.$$

(i) *Given $\alpha, \beta, \gamma \in \mathbf{Z}^n_+$, there exists $N_0 \in \mathbf{Z}^+$ such that for each $N \geq N_0$,*

$$\sup_{(x,y,z):S(x,y,z)>0} S(x,y,z)^N |D^\alpha_x D^\beta_y D^\gamma_z k(x,y,z)| < \infty$$

(ii) *Suppose that p has compact support in (ξ,η) uniformly in x. Then k is smooth, and given $\alpha,\beta,\gamma \in \mathbf{Z}^n_+$ and $N_0 \in \mathbf{Z}^+$, there exists $C > 0$ such that for all $x,y,z \in \mathbb{R}^n$ with $S(x,y,z) > 0$*

$$|D^\alpha_x D^\beta_y D^\gamma_z k(x,y,z)| \leq C(1 + S(x,y,z))^{-N}.$$

(iii) *Suppose that $m + M + 2n < 0$ for some $M \in \mathbf{Z}^+$. Then k is a bounded continuous function with bounded continuous derivatives of order $\leq M$.*

(iv) *Suppose that $m + M + 2n = 0$ for some $M \in \mathbf{Z}^+$. Then there exists a constant $C > 0$ such that for all $x,y,z \in \mathbb{R}^n$ with $S(x,y,z) > 0$,*

$$\sup_{|\alpha+\beta+\gamma|=M} |D^\alpha_x D^\beta_y D^\gamma_z k(x,y,z)| \leq C|\log|S(x,y,z)||.$$

(v) *Suppose that $m + M + 2n > 0$ for some $M \in \mathbf{Z}^+$. Then, given $\alpha,\beta,\gamma \in \mathbf{Z}^n_+$, there exists a positive constant C such that for all $x,y,z \in \mathbb{R}^n$ with $S(x,y,z) > 0$,*

$$\sup_{|\alpha+\beta+\gamma|=M} |\partial^\alpha_x \partial^\beta_y \partial^\gamma_z k(x,y,z)| \leq C S(x,y,z)^{-(m+M+2n)/\rho}.$$

Proof. Given a bilinear symbol $p(x,\xi,\eta)$ with $x,\xi,\eta \in \mathbb{R}^n$, set $X = (x_1,x_2) \in \mathbb{R}^{2n}$, $\zeta = (\xi,\eta) \in \mathbb{R}^{2n}$ and define the linear symbol P in $\mathbb{R}^{2n}$ as

$$P(X,\zeta) = p\left(\frac{x_1+x_2}{2}, \xi, \eta\right).$$

It follows easily that if $p \in BS^m_{\rho,\delta}(\mathbb{R}^n)$, for some $m \in \mathbb{R}$ and $\rho,\delta \in [0,1]$, then $P \in S^m_{\rho,\delta}(\mathbb{R}^{2n})$. Indeed, given $\alpha = (\alpha_1,\alpha_2), \beta = (\beta_1,\beta_2) \in \mathbf{Z}^{2n}_+$

$$\left|\partial^\alpha_X \partial^\beta_\zeta P(X,\zeta)\right| = \left|\left(\frac{1}{2}\right)^{|\alpha_1+\alpha_2|} (\partial^{\alpha_1+\alpha_2}_x \partial^{\beta_1}_\xi \partial^{\beta_2}_\eta p)\left(\frac{x_1+x_2}{2}, \xi, \eta\right)\right|$$

$$\leq C_{\alpha,\beta} \left(\frac{1}{2}\right)^{|\alpha_1+\alpha_2|} (1 + |\xi| + |\eta|)^{m-\rho|\beta_1+\beta_2|+\delta|\alpha_1+\alpha_2|}$$

$$= C_{\alpha,\beta} \left(\frac{1}{2}\right)^{|\alpha_1+\alpha_2|} (1 + |\zeta|)^{m-\rho|\beta|+\delta|\alpha|}.$$

In $\mathbb{R}^{2n}$, for the associated linear operator T_P we now have

$$
\begin{aligned}
T_P(F)(X) &= \int_{\mathbb{R}^{2n}} P(X,\zeta)e^{iX\cdot\zeta}\widehat{F}(\zeta)d\zeta \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} p\left(\frac{x_1+x_2}{2},\xi,\eta\right)e^{ix_1\cdot\xi}e^{ix_2\cdot\eta}\widehat{F}(\xi,\eta)\,d\xi d\eta,
\end{aligned}
$$

and also

$$
T_P(F)(X) = \int_{\mathbb{R}^{2n}} K(X,Y)F(Y)dY,
$$

where

$$
K(X,Y) = \mathcal{F}_{2n}(P(X,\cdot))(Y-X), \quad X,Y \in \mathbb{R}^{2n},
$$

and $\mathcal{F}_{2n}$ denotes the Fourier transform in $\mathbb{R}^{2n}$. Next, for the bilinear symbol p we write

$$
\begin{aligned}
T_p(f,g)(x) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} p(x,\xi,\eta)e^{ix\cdot(\xi+\eta)}\hat{f}(\xi)\hat{g}(\eta)d\xi d\eta \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} p(x,\xi,\eta)e^{ix\cdot\xi}e^{ix\cdot\eta}\widehat{f\otimes g}(\xi,\eta)d\xi d\eta \\
&= T_P(f\otimes g)(x,x) \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K((x,x),(y,z))(f\otimes g)(y,z)dydz \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K((x,x),(y,z))f(y)g(z)dydz.
\end{aligned}
$$

Therefore, the distributional bilinear kernel $k(x,y,z)$ of the bilinear operator T_p is given by

$$
k(x,y,z) = K((x,x),(y,z)), \quad x,y,z \in \mathbb{R}^n,
$$

where $K(X,Y)$ is the distributional linear kernel associated to the linear operator T_P in $\mathbb{R}^{2n}$. Finally, the pointwise estimates for linear kernels associated to symbols in $S^m_{\rho,\delta}(\mathbb{R}^{2n})$ in [1, Theorem 1.1] imply the desired pointwise estimates for the bilinear kernel $k(x,y,z)$. $\qquad\square$

6. An $L^2 \times W_0^{s,\infty} \to L^2$ boundedness property.

If $\sigma \in BS^0_{\rho,\delta}$, by freezing g, $T_\sigma(\cdot,g)$ can be regarded as a linear pseudodifferential operator (with symbol depending on g), that is,

$$
T_\sigma(f,g)(x) = \int_\xi \sigma_g(x,\xi)\hat{f}(\xi)e^{i\xi x}\,d\xi,
$$

where

$$
\sigma_g(x,\xi) = \int_\eta \sigma(x,\xi,\eta)\hat{g}(\eta)e^{i\eta x}\,d\eta.
$$

Moreover, the well-known L^2 boundedness of a linear pseudodifferential operator asserts that if $\tau \in S^0_{\rho,\delta}$, $0 \leq \delta \leq \rho \leq 1$, $\delta < 1$, there exist constants C_0 and $k \in \mathbb{N}$ (independent of τ) such that

$$\|T_\tau(u)\|_{L^2} \leq C_0 \left|\tau\right|_k \|u\|_{L^2}, \quad u \in \mathcal{S}(\mathbb{R}^n),$$

where

$$\left|\tau\right|_k = \max_{|\alpha|,|\beta| \leq k} \sup_{x,\xi} \left|\partial_x^\alpha \partial_\xi^\beta \tau(x,\xi)\right| (1+|\xi|)^{-\delta|\alpha|+\rho|\beta|}.$$

In fact, k can be taken equal to $[n/2]+1$, see [14, p. 30].

Theorem 7. *Let $\sigma \in BS^0_{\rho,\delta}$, $0 \leq \delta \leq \rho \leq 1$, $\delta < 1$. Then*

$$T_\sigma : L^2 \times W^{s,\infty}_0 \to L^2,$$

where s is any integer satisfying

(6.1) $$s > \frac{[n/2]+1}{1-\delta} + n.$$

Moreover, if $g \in C^s_c(\mathbb{R}^n)$ then $\sigma_g \in S^0_{\rho,\delta}$, and

$$\left|\sigma_g\right|_{[n/2]+1} \lesssim \|g\|_{W^{s,\infty}_0} := \sup_{|\gamma| \leq s} \|D^\gamma g\|_{L^\infty}$$

with a constant depending only on the $BS^0_{\rho,\delta}$-norm of σ up to order $n+2$.

Remark 6.1. Note that Theorem 7 includes the case $0 \leq \rho = \delta < 1$ and, in particular, the case $\rho = \delta = 0$, where, as pointed out in the introduction, the mapping from $L^2(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ fails.

Proof. Let $g \in C^s_c(\mathbb{R}^n)$ and $\sigma \in BS^0_{\rho,\delta}$, $0 \leq \delta \leq \rho \leq 1$, $\delta < 1$. We assume σ has compact support so all calculations below can be justified. However, all constants are independent of the support of σ and a limiting argument proves the result when σ does not have compact support (see [7]).

Fix multiindices α and β such that $|\alpha|, |\beta| \leq [n/2]+1$. Define $A = 1 + |\xi|$ and let $l_0 \in \mathbb{N}$ (to be chosen later) and $P_{l_0} = \{\gamma = (\gamma_1, \cdots, \gamma_n) : \gamma_i \text{ is even and } |\gamma| = 2j, j = 0, \ldots, l_0\}$. We have,

$$\partial_x^\alpha \partial_\xi^\beta \sigma_g(x,\xi) = \sum_{\gamma \leq \alpha} c_{\gamma,\alpha} \int_z \int_y \partial_x^\gamma \partial_\xi^\beta \sigma(x,\xi,z) z^{\alpha-\gamma} e^{izy} g(x-y) \, dydz$$

$$= \sum_{\gamma \leq \alpha} c_{\gamma,\alpha} \int_z \int_y (1 + A^{2\delta}(-\Delta_z))^{l_0} (\partial_x^\gamma \partial_\xi^\beta \sigma(x,\xi,z) z^{\alpha-\gamma}) \frac{e^{izy} g(x-y)}{(1+A^{2\delta}|y|^2)^{l_0}} \, dydz$$

$$= \sum_{\gamma \leq \alpha, \theta \in P_{l_0}} c_{\gamma,\alpha,\theta} \int_z \int_y A^{\delta|\theta|} \partial_z^\theta (\partial_x^\gamma \partial_\xi^\beta \sigma(x,\xi,z) z^{\alpha-\gamma}) \frac{e^{izy} g(x-y)}{(1+A^{2\delta}|y|^2)^{l_0}} \, dydz$$

$$= \sum_{\substack{\gamma \leq \alpha, \theta \in P_{l_0} \\ \omega \leq \min\{\theta, \alpha-\gamma\}}} c_{\gamma,\alpha,\theta,\omega} \int_z \int_y A^{\delta|\theta|} \partial_x^\gamma \partial_\xi^\beta \partial_z^{\theta-\omega} \sigma(x,\xi,z) z^{\alpha-\gamma-\omega} \frac{e^{izy} g(x-y)}{(1+A^{2\delta}|y|^2)^{l_0}} \, dydz.$$

Fix $\gamma \leq \alpha$, $\theta \in P_{l_0}$, $\omega \leq \min\{\theta, \alpha - \gamma\}$ and set

$$p(x, \xi) = \int_z \int_y A^{\delta|\theta|} \partial_x^\gamma \partial_\xi^\beta \partial_z^{\theta - \omega} \sigma(x, \xi, z) z^{\alpha - \gamma - \omega} \frac{e^{izy} g(x - y)}{(1 + A^{2\delta} |y|^2)^{l_0}} \, dydz.$$

Define the sets

$$\Omega_1 = \{z : |z| \leq \tfrac{A^\delta}{2}\}, \quad \Omega_2 = \{z : \tfrac{A^\delta}{2} \leq |z| \leq \tfrac{A}{2}\}, \quad \Omega_3 = \{z : |z| \geq \tfrac{A}{2}\}.$$

We then have

$$p(x, \xi) = \int_{\Omega_1} \int_y \cdots + \int_{\Omega_2} \int_y \cdots + \int_{\Omega_3} \int_y \cdots := I_1 + I_2 + I_3.$$

Note that

$$A \leq 1 + |\xi| + |z| \leq 2A, \qquad z \in \Omega_1 \cup \Omega_2$$

and that

$$|z| \leq 1 + |\xi| + |z| \leq 2|z|, \qquad z \in \Omega_3$$

Estimation for I_1. Choose l_0 such that $2l_0 > n$. Then

$$|I_1| \lesssim A^{\delta|\theta|} \|g\|_{L^\infty} \int_y \int_{|z| \leq \frac{A^\delta}{2}} (1 + |\xi| + |z|)^{\delta|\gamma| - \rho(|\beta| + |\theta| - |\omega|)} \frac{|z|^{|\alpha| - |\gamma| - |\omega|}}{(1 + A^{2\delta} |y|^2)^{l_0}} \, dzdy$$

$$\sim A^{\delta|\theta|} \|g\|_{L^\infty} A^{\delta|\gamma| - \rho(|\beta| + |\theta| - |\omega|)} A^{-\delta n} A^{\delta(|\alpha| - |\gamma| - |\omega| + n)}$$

$$= A^{(\delta - \rho)(|\theta| - |w|)} A^{\delta|\alpha| - \rho|\beta|} \|g\|_{L^\infty}$$

$$\leq A^{\delta|\alpha| - \rho|\beta|} \|g\|_{L^\infty}.$$

Note that in the last inequality we have used that $\delta \leq \rho$ and $|\theta| - |w| \geq 0$.

Estimation for I_2. Let $l \in \mathbb{N}$ to be chosen later. We have

$$I_2 = \int_{\Omega_2} \int_y A^{\delta|\theta|} \partial_x^\gamma \partial_\xi^\beta \partial_z^{\theta - \omega} \sigma(x, \xi, z) z^{\alpha - \gamma - \omega} \frac{g(x - y)}{(1 + A^{2\delta} |y|^2)^{l_0}} \frac{(-\Delta_y)^l (e^{izy})}{|z|^{2l}} \, dydz$$

$$= \int_{\Omega_2} A^{\delta|\theta|} \frac{z^{\alpha - \gamma - \omega}}{|z|^{2l}} \partial_x^\gamma \partial_\xi^\beta \partial_z^{\theta - \omega} \sigma(x, \xi, z) \int_y (-\Delta_y)^l \left(\frac{g(x - y)}{(1 + A^{2\delta} |y|^2)^{l_0}} \right) e^{izy} \, dy \, dz.$$

Now,

$$\left| (-\Delta_y)^l \left(\frac{g(x - y)}{(1 + A^{2\delta} |y|^2)^{l_0}} \right) \right| = \left| \sum_{\substack{|\mu| = 2l \\ \mu_i even, \nu \leq \mu}} C_{\nu\mu} \partial_y^\nu ((1 + A^{2\delta} |y|^2)^{-l_0}) \partial_y^{\mu - \nu} g(x - y) \right|$$

$$\leq \sum_{\substack{|\mu| = 2l \\ \mu_i even, \nu \leq \mu}} C_{\nu\mu l_0} \|D^{\mu - \nu} g\|_{L^\infty} A^{\delta|\nu|} (1 + A^{2\delta} |y|^2)^{-l_0}.$$

Therefore,

$$|I_2| \leq \int_{\Omega_2} A^{\delta|\theta|} \frac{|z^{\alpha-\gamma-\omega}|}{|z|^{2l}} \left| \partial_x^\gamma \partial_\xi^\beta \partial_z^{\theta-\omega} \sigma(x,\xi,z) \right| \sum_{\substack{|\mu|=2l \\ \mu_i even, \nu \leq \mu}} C_{\nu\mu l_0} \|D^{\mu-\nu}g\|_{L^\infty} A^{\delta|\nu|} A^{-\delta n} \, dz$$

$$\leq \sum_{\substack{|\mu|=2l \\ \mu_i even, \nu \leq \mu}} C_{\nu\mu l_0} \|D^{\mu-\nu}g\|_{L^\infty} A^{\delta(|\theta|+|\nu|-n)} \int_{\Omega_2} |z|^{|\alpha|-|\gamma|-|\omega|-2l} (A+|z|)^{\delta|\gamma|-\rho(|\beta|+|\theta|-|\omega|)} \, dz$$

$$\leq A^{\delta(|\theta|+2l-n)} \|g\|_{W_0^{2l,\infty}} A^{\delta|\gamma|-\rho(|\beta|+|\theta|-|\omega|)} A^{\delta(|\alpha|-|\gamma|-|\omega|-2l+n)},$$

where we have used that $A + |z| \sim A$ on Ω_2 and l has been chosen so that

(6.2) $$|\alpha| - |\gamma| - |\omega| - 2l + n < 0.$$

Thus we obtain,

$$|I_2| \leq C\, A^{\delta(|\theta|+2l-n)} \|g\|_{W_0^{2l,\infty}} A^{\delta|\gamma|-\rho(|\beta|+|\theta|-|\omega|)} A^{\delta(|\alpha|-|\gamma|-|\omega|-2l+n)}$$

$$= C\, A^{\delta|\alpha|-\rho|\beta|} A^{(\delta-\rho)(|\theta|-|w|)} \|g\|_{W_0^{2l,\infty}}$$

$$\leq C\, A^{\delta|\alpha|-\rho|\beta|} \|g\|_{W_0^{2l,\infty}},$$

since $(\delta - \rho)(|\theta| - |w|) \leq 0$.

Estimation for I_3. Here we impose some extra conditions to l above. As in the estimation for B_2 we have

$$|I_3| \leq \sum_{\substack{|\mu|=2l \\ \mu_i even, \nu \leq \mu}} C_{\nu\mu l_0} \|D^{\mu-\nu}g\|_{L^\infty} A^{\delta(|\theta|+|\nu|-n)} \int_{\Omega_3} |z|^{|\alpha|-|\gamma|-|\omega|-2l} (A+|z|)^{\delta|\gamma|-\rho(|\beta|+|\theta|-|\omega|)} \, dz.$$

Using that $A + |z| \sim |z|$ in Ω_3 we have

$$|I_3| \leq C\, \|g\|_{W_0^{2l,\infty}} A^{\delta(|\theta|+2l-n)} \int_{\Omega_3} |z|^{|\alpha|-|\gamma|-|\omega|-2l+\delta|\gamma|-\rho(|\beta|+|\theta|-|\omega|)} \, dz.$$

Now, let us choose l as in the estimation of I_2 and satisfying

$$|\alpha| - |\gamma| - |\omega| - 2l + \delta\,|\gamma| - \rho(|\beta| + |\theta| - |\omega|) + n < 0.$$

For example, any choice of l such that $2l > |\alpha| + n$ satisfies the inequality above. Then,

$$|I_3| \leq C\, A^{\delta(|\theta|+2l-n)} A^{|\alpha|-|\gamma|-|\omega|-2l+\delta|\gamma|-\rho(|\beta|+|\theta|-|\omega|)+n} \|g\|_{W_0^{2l,\infty}}$$

$$= C\, A^{\delta|\gamma|-\rho|\beta|} A^{(\delta-1)(2l-n)+|\alpha|-|\gamma|} A^{(\delta-\rho)|\theta|} A^{(\rho-1)|\omega|} \|g\|_{W_0^{2l,\infty}}.$$

Since $\delta < 1$, we can choose l sufficiently large so that

(6.3) $$(\delta - 1)(2l - n) + |\alpha| - |\gamma| \leq 0$$

and since $\delta - \rho \leq 0$, $\rho - 1 \leq 0$ and $|\gamma| \leq |\alpha|$ we obtain

$$|I_3| \leq C\, A^{\delta|\alpha|-\rho|\beta|} \|g\|_{W_0^{2l,\infty}}.$$

Finally, by choosing $l = s/2$, with s as in (6.1), we guarantee that both conditions (6.2) and (6.3) are satisfied, and the proof is complete. $\square$

Corollary 8. *Let $m \geq 0$ and $\sigma \in BS_{\rho,\delta}^m$, $0 \leq \delta \leq \rho \leq 1$, $\delta < 1$. Then, if s is any integer satisfying* (6.1)*, the following fractional Leibniz rule-type inequality holds true*

$$(6.4) \quad \|T_\sigma(f,g)\|_{L^2} \leq C \left(\|f\|_{W^{m,2}} \|g\|_{W_0^{s,\infty}} + \|f\|_{W_0^{s,\infty}} \|g\|_{W^{m,2}} \right), \quad f, g \in C_c^\infty(\mathbb{R}^n).$$

Proof. Corollary 8 follows from Theorem 7 and composition with Bessel potentials of order m, along the lines of Theorem 2 in [5]. We only need to notice that, if $\sigma \in BS_{\rho,\delta}^m$ and ϕ is a C^∞ function on $\mathbb{R}$ such that $0 \leq \phi \leq 1$, $\text{supp}(\phi) \subset [-2, 2]$ and $\phi(r) + \phi(1/r) = 1$ on $[0, \infty)$, then, the symbols σ_1 and σ_2 defined by

$$\sigma_1(x, \xi, \eta) = \sigma(x, \xi, \eta)\phi\left(\frac{1 + |\xi|^2}{1 + |\eta|^2}\right)(1 + |\xi|^2)^{-m/2}$$

and

$$\sigma_2(x, \xi, \eta) = \sigma(x, \xi, \eta)\phi\left(\frac{1 + |\eta|^2}{1 + |\xi|^2}\right)(1 + |\eta|^2)^{-m/2}$$

satisfy $\sigma_1, \sigma_2 \in BS_{\rho,\delta}^0$, and the corresponding operators T_σ, T_{σ_1}, and T_{σ_2} are related through

$$T_\sigma(f,g) = T_{\sigma_1}(J^m f, g) + T_{\sigma_2}(f, J^m g),$$

where J^m denotes the linear Fourier multiplier with symbol $(1 + |\xi|^2)^{m/2}$. $\square$

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